

# A NON-UNIFORM DISTRIBUTION PROPERTY OF MOST ORBITS, IN CASE THE $3x + 1$ CONJECTURE IS TRUE

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ABSTRACT. Let  $T(n) = \begin{cases} 3n+1 & (n \text{ odd}) \\ \frac{n}{2} & (n \text{ even}) \end{cases}$  ( $n \in \mathbb{Z}$ ). We call “the orbit of the integer  $n$ ”, the set

$$\mathcal{O}_n := \{m \in \mathbb{Z} : \exists k \geq 0, m = T^k(n)\}$$

and we put  $c_i(n) := \#\{m \in \mathcal{O}_n : m \equiv i \pmod{18}\}$ . Let  $W$  be the set of the integers whose orbit contains 1 and is, in the following sense, approximately well distributed modulo 18 between the six elements of the set  $I := \{1, 5, 7, 11, 13, 17\}$  (the elements of  $\{1, \dots, 18\}$  that are odd and not divisible by 3). More precisely:

$$W := \left\{ n \in \mathbb{Z} : \exists k \geq 0, T^k(n) = 1 \text{ and } \forall i \in I, \frac{c_i(n)}{\sum_{i \in I} c_i(n)} \leq \frac{1}{6} + 0.0215 \right\}.$$

We prove that  $W \cap \mathbb{N}$  has density 0 in  $\mathbb{N}$ . Consequently, if the  $3x + 1$  conjecture is true, most of the positive integers  $n$  satisfy

$$\frac{\max_{i \in I} c_i(n)}{\sum_{i \in I} c_i(n)} > \frac{1}{6} + 0.0215.$$

*Dedicated to the memory of Pierre Liardet*

## 1. INTRODUCTION

As it can be seen in one example given by Lagarias, if we chose a large integer (for instance the one of figure 2 in <http://www.ams.org/bookstore/pspdf/mbk-78-prev.pdf>), in general its orbit under the transform  $T := n \mapsto \begin{cases} 3n+1 & (n \text{ odd}) \\ \frac{n}{2} & (n \text{ even}) \end{cases}$  contains about two times less odd numbers than even numbers, due to the fact that  $3n + 1$  is even for any odd  $n$ . This figure shows that the orbit of the integer  $100 \lfloor \pi \cdot 10^{35} \rfloor$  has length about 900 and, as expected, about 300 odd and 600 even elements, because  $100 \lfloor \pi \cdot 10^{35} \rfloor \cdot \frac{3^{300}}{2^{600}} \approx 1$ . Fortunately, the method we use to prove the following theorem can't be used to contradict this property.

**Theorem 1.** *We put*

$$\mathcal{O}_n := \{m \in \mathbb{Z} : \exists k \geq 0, m = T^k(n)\} \quad (\text{orbit of the integer } n),$$

$$c_i(n) := \#\{m \in \mathcal{O}_n : m \equiv i \pmod{18}\} \quad (\text{finite or infinite}),$$

$$I := \{1, 5, 7, 11, 13, 17\},$$

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$$W := \left\{ n \in \mathbb{Z} : \exists k \geq 0, T^k(n) = 1 \text{ and } \forall i \in I, \frac{c_i(n)}{\sum_{i \in I} c_i(n)} \leq \frac{1}{6} + 0.0215 \right\}.$$

We have for any  $N$  large enough

$$\#W \cap \{1, \dots, N\} \leq N^{0.9999}.$$

Of course this theorem remains true if we replace the condition  $T^k(n) = 1$  by  $T^k(n) = n_0$ , where  $n_0 \in \mathbb{Z} \setminus \{0\}$  is fixed. In case  $n_0 < 0$  we replace the interval  $\{1, \dots, N\}$  by  $\{-N, \dots, -1\}$ .

To prove this theorem we use the same method as Krasilov and Lagarias [4], it consists in describing the set of the antecedents of 1 by the powers of  $T$ . See also [1, 2, 3, 5, 6, 7, 8].

**Remark 2.** To give a numerical example we consider the orbit of each of the integers  $n \in \{1, \dots, 26\}$  and we compute  $c_i := \sum_{n=1}^{26} c_i(n)$ :

$$(c_1, \dots, c_{18}) = (28, 41, 5, 49, 22, 4, 5, 37, 2, 23, 10, 2, 11, 4, 1, 47, 13, 1).$$

As expected,  $\sum_{i \text{ odd}} c_i = 97$  is close to the half of  $\sum_{i \text{ even}} c_i = 208$ . Among the  $c_i$  with  $i$  odd,  $c_7 = 5$  is smaller than  $c_1 = 28$ ,  $c_5 = 22$ ,  $c_{11} = 10$ ,  $c_{13} = 11$  and  $c_{17} = 13$ . The proof of the theorem allows to see, in the general case when  $n \in \{1, \dots, N\}$ , why  $c_7$  is smaller than  $c_1, c_5, c_{11}, c_{13}, c_{17}$ . On the other hand the  $c_i$  for  $i$  a multiple of 3 are small for an obvious reason:  $\frac{3n+1}{2^k}$  is never a multiple of 3.

## 2. THE NOTATIONS WE USE TO DESCRIBE THE SET OF THE ANTECEDENTS OF 1

Instead of  $T$  we use the transform defined by Sinai in [9], that we call  $S$ :

$$\begin{aligned} S : \square &\rightarrow \square \\ \square &:= \{n \in \mathbb{Z} : n \text{ odd and } n \notin 3\mathbb{Z}\} = \{1, 5, 7, 11, 13, 17\} + 18\mathbb{Z} \\ S(n) &:= \frac{3n+1}{2^k}, \quad k \in \mathbb{N}. \end{aligned}$$

The antecedents of 1 by  $S$  are the integers

$$n_1 = \frac{1}{3}(2^{\varepsilon_1} - 1) \tag{1}$$

that belong to  $\square$ ; this is equivalent to  $\varepsilon_1 \in \{2, 4\} \bmod 6$ . Let now  $n_\alpha, n_{\alpha-1}, \dots, n_1$  be some integers such that

$$n_\alpha \xrightarrow{S} n_{\alpha-1} \xrightarrow{S} \dots \xrightarrow{S} n_1 \xrightarrow{S} n_0 := 1.$$

For any  $0 \leq j < \alpha$  there exists  $\varepsilon_{j+1} \in \mathbb{N}$  such that

$$n_{j+1} = \frac{1}{3}(2^{\varepsilon_{j+1}} n_j - 1). \tag{2}$$

One has  $n_{j+1} \in \square$ , and this is equivalent to  $2^{\varepsilon_{j+1}} n_j - 1 \in 3\mathbb{N} \setminus 9\mathbb{N}$ . This means that, when we know the value of  $n_j$ , or equivalently when we know  $\varepsilon_1, \dots, \varepsilon_j$ , the positive integer  $\varepsilon_{j+1}$  must satisfy the conditions:

$$\begin{aligned} \text{if } n_j &\equiv 1 \bmod{18}, & \varepsilon_{j+1} &\in \{2, 4\} \bmod{6} \\ \text{if } n_j &\equiv 5 \bmod{18}, & \varepsilon_{j+1} &\in \{3, 5\} \bmod{6} \\ \text{if } n_j &\equiv 7 \bmod{18}, & \varepsilon_{j+1} &\in \{4, 6\} \bmod{6} \\ \text{if } n_j &\equiv 11 \bmod{18}, & \varepsilon_{j+1} &\in \{1, 3\} \bmod{6} \\ \text{if } n_j &\equiv 13 \bmod{18}, & \varepsilon_{j+1} &\in \{2, 6\} \bmod{6} \\ \text{if } n_j &\equiv 17 \bmod{18}, & \varepsilon_{j+1} &\in \{1, 5\} \bmod{6} \end{aligned} \tag{3}$$

(notice that the case  $n_j \equiv 7 \pmod{18}$  gives the largest values:  $\varepsilon_{j+1} \geq 4$  and  $n_{j+1} \geq \frac{1}{3}(16n_j - 1)$ ). So all the antecedents of 1 by  $S^\alpha$  are obtained by the following formula, subject to the conditions (3):

$$n_\alpha = \frac{1}{3^\alpha} (2^{\varepsilon_1 + \dots + \varepsilon_\alpha} - 2^{\varepsilon_2 + \dots + \varepsilon_\alpha} 3^0 - \dots - 2^{\varepsilon_\alpha} 3^{\alpha-2} - 2^0 3^{\alpha-1}). \quad (4)$$

We give a first estimation of  $n_\alpha$ :

**Lemma 3.** *If  $\alpha \geq 2$  and  $\varepsilon_1 \neq 2$ ,*

$$\frac{2^{\varepsilon_1 + \dots + \varepsilon_\alpha}}{\alpha 3^\alpha} \leq n_\alpha \leq \frac{2^{\varepsilon_1 + \dots + \varepsilon_\alpha}}{3^\alpha}.$$

*Proof.* The upper bound is an immediate consequence of (4). The lower bound can be deduced from the straightforward equality:

$$3n+1 = 3^{1+\alpha_n} n \quad \text{with} \quad \alpha_n = \frac{1}{\log 3} \log \left( 1 + \frac{1}{3n} \right) \leq \frac{1}{3n}. \quad (5)$$

Indeed (2) and (5) imply

$$n_j \leq \frac{3^{1+\frac{1}{3n_{j+1}}}}{2^{\varepsilon_{j+1}}} n_{j+1}$$

hence

$$n_0 \leq \frac{3^{\alpha + \frac{1}{3n_1} + \dots + \frac{1}{3n_\alpha}}}{2^{\varepsilon_1 + \dots + \varepsilon_\alpha}} n_\alpha.$$

Now the  $n_j$  are distinct (no cycle between  $n_\alpha$  and 1, because (1) and the hypothesis  $\varepsilon_1 \neq 2$  imply  $n_1 \neq 1$ ), and consequently  $\frac{1}{n_1} + \dots + \frac{1}{n_\alpha} \leq \frac{1}{1} + \dots + \frac{1}{\alpha} \leq 1 + \log \alpha$ . If  $\alpha \geq 2$ , the inequality  $3^{\frac{1}{3n_1} + \dots + \frac{1}{3n_\alpha}} \leq \alpha$  and the lemma follow.  $\square$

Here we give an indexation and a new lower bound for  $n_\alpha$ :

**Lemma 4.** *There exists a one-to-one map*

$$\mathbf{n} : (\mathbb{N} \setminus \{1\}) \times \mathbb{N}^{\alpha-1} \leftrightarrow \Pi_\alpha := \{n \in \Pi : S^\alpha(n) = 1 \neq S^{\alpha-1}(n)\}$$

*such that – for any  $(i_1, \dots, i_\alpha) \in \mathbb{N} \setminus \{1\} \times \mathbb{N}^{\alpha-1}$*

$$\begin{aligned} \mathbf{n}(i_1, \dots, i_\alpha) &\geq \frac{2^{3(i_1 + \dots + i_\alpha) - c(\mathbf{n}(i_1, \dots, i_{\alpha-1})) + \alpha'(i_1, \dots, i_\alpha)}}{\alpha 3^\alpha}, \quad \text{where} \\ c(n) &:= 2c_1(n) + c_5(n) + 3c_{11}(n) + 2c_{13}(n) + 3c_{17}(n) \quad (n \in \mathbb{Z}), \\ \alpha'(i_1, \dots, i_\alpha) &:= \#\{1 \leq j \leq \alpha : i_j \text{ odd}\}. \end{aligned} \quad (6)$$

*Proof.* We define  $\mathbf{n}(i_1, \dots, i_\alpha)$  by induction on  $\alpha$ . When  $\alpha = 1$ , according to (1) the antecedents of 1 by  $S$ , distinct from 1, are the following integers indexed by  $i_1 \geq 2$ :

$$\mathbf{n}(i_1) := \frac{1}{3} \left( 2^{\varepsilon_1(i_1)} - 1 \right) \quad \text{where } \forall i, \quad \varepsilon_1(i) := 3i - 1 \text{ (} i \text{ odd) or } 3i - 2 \text{ (} i \text{ even)}. \quad (7)$$

Suppose now that  $\mathbf{n}(i_1, \dots, i_j)$  (antecedent of 1 by  $S^j$  and not by  $S^{j-1}$ ) is already defined for any  $(i_1, \dots, i_j) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}^{j-1}$ . We denote by  $0 < \varepsilon(i_1, \dots, i_j, 1) < \varepsilon(i_1, \dots, i_j, 2) < \dots$  ( $i \in \mathbb{N}$ ) the possible values of  $\varepsilon_{j+1}$  in (3); the antecedents of  $\mathbf{n}(i_1, \dots, i_j)$  by  $S$  are

$$\mathbf{n}(i_1, \dots, i_j, i) := \frac{1}{3} \left( 2^{\varepsilon(i_1, \dots, i_j, i)} \mathbf{n}(i_1, \dots, i_j) - 1 \right) \quad (i \in \mathbb{N}).$$

We obtain in this way all the antecedents of 1 by  $S^{j+1}$  that are not antecedents of 1 by  $S^j$ . With this notations, the conditions in (3) are equivalent to

$$\begin{aligned} \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 1, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i - 1 \quad (i \text{ odd}) \quad \text{or} \quad 3i - 2 \quad (i \text{ even}) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 5, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i \quad (i \text{ odd}) \quad \text{or} \quad 3i - 1 \quad (i \text{ even}) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 7, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i + 1 \quad (i \text{ odd}) \quad \text{or} \quad 3i \quad (i \text{ even}) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 11, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i - 2 \quad (i \text{ odd}) \quad \text{or} \quad 3i - 3 \quad (i \text{ even}) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 13, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i - 1 \quad (i \text{ odd}) \quad \text{or} \quad 3i \quad (i \text{ even}) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 17, \quad \varepsilon(i_1, \dots, i_j, i) &= 3i - 2 \quad (i \text{ odd}) \quad \text{or} \quad 3i - 1 \quad (i \text{ even}). \end{aligned}$$

Setting  $r(i) := \begin{cases} 1 & (i \text{ odd}) \\ 0 & (i \text{ even}) \end{cases}$  (remainder of  $n$  modulo 2), we have for any  $i \in \mathbb{N}$

$$\begin{aligned} \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 1, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 2 + r(i) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 5, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 1 + r(i) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 7, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 0 + r(i) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 11, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 3 + r(i) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 13, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 2 + r(i) \\ \text{if } \mathbf{n}(i_1, \dots, i_j) \equiv 17, \quad \varepsilon(i_1, \dots, i_j, i) &\geq 3i - 3 + r(i). \end{aligned} \tag{8}$$

We consider the formula (7), and the formulas (8) for  $j = 1, \dots, \alpha - 1$ : they depend on the value modulo 18 of the integers  $1, \mathbf{n}(i_1), \dots, \mathbf{n}(i_1, \dots, i_{\alpha-1})$  respectively. In other words, these formulas depend on the orbit of  $\mathbf{n}(i_1, \dots, i_{\alpha-1})$  by  $S$ . Using Lemma 3 and the definitions of  $c(n)$  and  $\alpha'(i_1, \dots, i_\alpha) = \sum_{j=0}^{\alpha} r(i_j)$  we deduce

$$\mathbf{n}(i_1, \dots, i_\alpha) \geq \frac{2^{\sum_{j=1}^{\alpha} \varepsilon(i_1, \dots, i_j)}}{\alpha 3^\alpha} \geq \frac{2^{3(i_1 + \dots + i_\alpha) - c(\mathbf{n}(i_1, \dots, i_{\alpha-1})) + \alpha'(i_1, \dots, i_\alpha)}}{\alpha 3^\alpha}.$$

□

### 3. A FIRST BOUND FOR $\#W \cap \{1, \dots, N\}$

In the following lemma we specify how to obtain all the antecedents of 1 by the powers of  $S$  or  $T$ .

**Lemma 5.** (i) *The set of the antecedents of 1 by the powers of  $S$  (resp. by the powers of  $T$ ), namely*

$$\mathcal{S} := \{n \in \mathbb{N} : \exists \alpha \geq 0, S^\alpha(n) = 1\} \quad (\text{resp. } \mathcal{T} := \{n \in \mathbb{N} : \exists k \geq 0, T^k(n) = 1\}),$$

can also be defined by

$$\mathcal{S} = \bigcup_{\alpha \geq 0} \Pi_\alpha \quad (\text{where } \Pi_0 := \{1\}) \quad \text{and} \quad \mathcal{T} = \mathbb{N} \cap \bigcup_{i \geq 0} \bigcup_{j \geq 1} \frac{2^i}{3} (2^j \mathcal{S} - 1).$$

(ii) *If  $n \in \mathcal{S}$  there exist  $\alpha \geq 0, i'_1, \dots, i'_\alpha \geq 1$  and  $A \subset \{1, \dots, \alpha\}$  such that*

$$n = \mathbf{n}(i_1, \dots, i_\alpha) \quad \text{with } i_j = \begin{cases} 2i'_j - 1 & \text{if } j \in A \\ 2i'_j & \text{else} \end{cases} \tag{9}$$

and  $i'_1 \neq 1$  if  $1 \in A$ . If  $n = \mathbf{n}(i_1, \dots, i_\alpha)$  belongs to  $W$ ,

$$i'_1 + \dots + i'_\alpha - \frac{11}{6} \left( \frac{1}{6} + 0.0215 \right) (\alpha + 1) - \frac{1}{3} \#A - \frac{\log \alpha}{6 \log 2} - \alpha \frac{\log 3}{6 \log 2} \leq \frac{\log n}{6 \log 2}. \tag{10}$$

*Proof.* (i) The first relation is obvious and the second follows from the fact that the orbit of any  $n \in \mathbb{N}$  by the transformation  $T$ , begins by

$$n \xrightarrow{T^i \ (i \geq 0)} \frac{n}{2^i} \xrightarrow{T} 3\frac{n}{2^i} + 1 \xrightarrow{T^j \ (j \geq 1)} \frac{3\frac{n}{2^i} + 1}{2^j} \in \Pi \quad (\text{or } \in \mathcal{S}, \text{ if } n \in \mathcal{T}).$$

(ii) (9) is a consequence of Lemma 4. For any  $n \in W$ ,

$$c(n) \leq 11 \left( \frac{1}{6} + 0.0215 \right) \sum_{i \in I} c_i(n) = 11 \left( \frac{1}{6} + 0.0215 \right) \#\{m \in \mathcal{O}_n : m \text{ odd}\}. \quad (11)$$

Now if  $n = \mathbf{n}(i_1, \dots, i_\alpha)$  there are  $\alpha + 1$  odd integers in  $\mathcal{O}_n$ , so (10) follows from (6) and (11).  $\square$

**Lemma 6.** *There exists a constant  $K$  such that*

$$\#W \cap \{1, \dots, N\} \leq K(\log N)^K \max_{(\alpha', \alpha'') \in A(N)} \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!}$$

where  $\alpha''' = \alpha'''(N, \alpha', \alpha'')$  is defined by

$$\alpha'''(N, \alpha', \alpha'') := \lfloor 0.2405 \log N + 0.345 - 0.05749 \alpha' - 0.39083 \alpha'' \rfloor$$

and  $A(N) := \{(\alpha', \alpha'') \in (\mathbb{N} \cup \{0\})^2 : \alpha'''(N, \alpha', \alpha'') \geq 0\}$ .

*Proof.* We use the one-to-one map  $\mathbf{n} : (\mathbb{N} \setminus \{1\}) \times \mathbb{N}^{\alpha-1} \leftrightarrow \Pi_\alpha$  defined in Lemma 4, and the notation

$$\begin{aligned} A(i_1, \dots, i_\alpha) &:= \{1 \leq j \leq \alpha : i_j \text{ odd}\} \\ \Pi_{\alpha, \alpha'} &:= \{n = \mathbf{n}(i_1, \dots, i_\alpha) \in \Pi_\alpha : \#A(i_1, \dots, i_\alpha) = \alpha'\}. \end{aligned}$$

Now the nonnegative integers  $N \in \mathbb{N}$ ,  $\alpha, \alpha', \alpha'' \geq 0$  are fixed with  $\alpha = \alpha' + \alpha''$ . Assume for instance that  $\alpha \geq 10^{10}$ : then we have  $\frac{\log \alpha}{\alpha} \leq 10^{-8}$ . We use the notations of Lemma 5 (ii); we deduce from (10) that, if there exists at least one element  $n = \mathbf{n}(i_1, \dots, i_\alpha) \in \Pi_{\alpha, \alpha'} \cap W \cap \{1, \dots, N\}$ ,

$$i'_1 + \dots + i'_\alpha - 0.345(\alpha + 1) - 0.33334 \alpha' - 0.26417 \alpha \leq 0.2405 \log n. \quad (12)$$

This inequality is equivalent to

$$i'_1 + \dots + i'_\alpha \leq \alpha + \alpha'''(n, \alpha', \alpha'').$$

This last inequality with  $\alpha \leq i'_1 + \dots + i'_\alpha$  implies  $\alpha'''(n, \alpha', \alpha'') \geq 0$  and a fortiori  $\alpha'''(N, \alpha', \alpha'') \geq 0$ . So we have proved that, if the set  $\Pi_{\alpha, \alpha'} \cap W \cap \{1, \dots, N\}$  is not empty,  $(\alpha', \alpha'')$  belongs to  $A(N)$ .

Let us bound the number of elements of  $\Pi_{\alpha, \alpha'} \cap W \cap \{1, \dots, N\}$ . We can associate injectively to any  $n$  in this set, some integers  $i'_1, \dots, i'_\alpha$  such that

$$1 \leq i'_1 < i'_1 + i'_2 < \dots < i'_1 + \dots + i'_\alpha \leq \alpha + \alpha''' \quad (\text{where } \alpha''' = \alpha'''(N, \alpha', \alpha''))$$

and a subset  $A \subset \{1, \dots, \alpha\}$  of cardinality  $\alpha'$ , such that

$$A(i_1, \dots, i_\alpha) = A, \text{ where } i_j = \begin{cases} 2i'_j - 1 & \text{if } n \in A \\ 2i'_j & \text{else.} \end{cases}$$

Consequently

$$\#\Pi_{\alpha, \alpha'} \cap W \cap \{1, \dots, N\} \leq \binom{\alpha + \alpha'''}{\alpha} \cdot \binom{\alpha}{\alpha'} = \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!}.$$

The inequality  $\alpha'''(N, \alpha', \alpha'') \geq 0$  implies  $\alpha \leq K_1 \log N$  with  $K_1$  constant, hence

$$\sum_{\alpha=10^{10}}^{\lfloor K_1 \log N \rfloor} \sum_{\alpha'=0}^{\alpha} \# \Pi_{\alpha, \alpha'} \cap W \cap \{1, \dots, N\} \leq (K_1 \log N)^2 \max_{(\alpha', \alpha'') \in A(N)} \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!}. \quad (13)$$

It remains to bound  $\sum_{\alpha=0}^{10^{10}-1} \# \Pi_{\alpha} \cap W \cap \{1, \dots, N\}$ . One can associate injectively to any  $n \in \Pi_{\alpha} \cap W \cap \{1, \dots, N\}$ , some positive integers  $\varepsilon_1, \dots, \varepsilon_{\alpha}$  such that (4) holds. According to Lemma 3 one has  $2^{\varepsilon_1 + \dots + \varepsilon_{\alpha}} \leq \alpha 3^{\alpha} N < 10^{10} 3^{10^{10}} N$ , hence any  $\varepsilon_j$  is bounded by  $K_2 \log N$  with  $K_2$  constant. Consequently

$$\sum_{\alpha=0}^{10^{10}-1} \# \Pi_{\alpha} \cap W \cap \{1, \dots, N\} \leq 10^{10} (K_2 \log N)^{10^{10}}. \quad (14)$$

From Lemma 5 (i),  $\mathcal{S}$  is the union of the  $\Pi_{\alpha}$  hence, from (13) and (14), there exists a constant  $K_3$  such that

$$\# \mathcal{S} \cap W \cap \{1, \dots, N\} \leq K_3 (\log N)^{K_3} \max_{(\alpha', \alpha'') \in A(N)} \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!}. \quad (15)$$

Let us bound now  $\# W \cap \{1, \dots, N\}$ . By Lemma 5 (i) any  $n \in \mathcal{T} \cap W \cap \{1, \dots, N\} = W \cap \{1, \dots, N\}$  can be written

$$n = \frac{2^i}{3} (2^j s - 1) \text{ with } i \geq 0, j \geq 1, s \in \mathcal{S} \cap W.$$

This implies  $s \leq 2N$ ,  $2^i \leq 3N$  and  $2^j \leq 3N + 1$ . So there are at most  $K_4 (\log N)^2$  possible values for the couple  $(i, j)$ , with  $K_4$  constant, and

$$\# W \cap \{1, \dots, N\} \leq K_4 (\log N)^2 \# \mathcal{S} \cap W \cap \{1, \dots, 2N\}. \quad (16)$$

The lemma follows from (15) and (16).  $\square$

#### 4. PROOF OF THE THEOREM

**Lemma 7.** *Let  $\ell(N) := 0.2405 \log N + 0.345$ . With the notations of Lemma 6, one has*

$$\max_{(\alpha', \alpha'') \in A(N)} \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!} \leq \max_{(x, y) \in T} \left( \frac{(x + y + z)^{x+y+z}}{x^x y^y z^z} \right)^{\ell(N)}$$

where  $z = 1 - 0.05749 x - 0.39083 y$  and  $T := \{(x, y) : x \geq 0, y \geq 0, z \geq 0\}$ .

*Proof.* Let  $(\alpha', \alpha'') \in A(N)$ , the reals

$$x = \frac{\alpha'}{\ell(N)} \quad \text{and} \quad y = \frac{\alpha''}{\ell(N)},$$

satisfy  $(x, y) \in T$ . By Corollary 9 one has

$$\frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'! \alpha''! \alpha'''!} \leq \frac{(\alpha' + \alpha'' + \alpha''')^{\alpha' + \alpha'' + \alpha'''}}{\alpha'^{\alpha'} \alpha''^{\alpha''} \alpha'''^{\alpha'''}}. \quad (17)$$

The map  $t \mapsto \frac{(\alpha' + \alpha'' + t)^{\alpha' + \alpha'' + t}}{\alpha'^{\alpha'} \alpha''^{\alpha''} t^t}$  is not decreasing because the derivative of its logarithm is  $\log(1 + \frac{\alpha' + \alpha''}{t}) \geq 0$ . Recall that  $\alpha''' = \alpha'''(N, \alpha', \alpha'')$  is the integral part of

$$\alpha''' = \alpha'''(N, \alpha', \alpha'') := \ell(N) - 0.0575 \alpha' - 0.39084 \alpha'',$$

so one has

$$\frac{(\alpha' + \alpha'' + \alpha''')^{\alpha' + \alpha'' + \alpha'''}}{\alpha'^{\alpha'} \alpha''^{\alpha''} \alpha'''^{\alpha'''}} \leq \frac{(\alpha' + \alpha'' + \alpha''')^{\alpha' + \alpha'' + \alpha'''}}{\alpha'^{\alpha'} \alpha''^{\alpha''} \alpha'''^{\alpha'''}}. \quad (18)$$

Lemma 7 results from (17) and (18) because the real  $z$ , as defined in this lemma, is equal to  $\frac{\alpha''''}{\ell(N)}$ .  $\square$

*End of the proof of the theorem.* We apply Lemma 10 to  $a = 0.05749$  and  $b = 0.39083$ : the function  $\varphi$  attains its maximum in the interior of  $T$ , let  $(x_0, y_0)$  be a point where  $\varphi$  is maximal and let  $z_0 = 1 - ax_0 - by_0$ . Since  $\varphi$  is differentiable on the interior of  $T$ , the partial derivatives are null at  $(x_0, y_0)$ :

$$\begin{cases} (1-a)\log(x_0 + y_0 + z_0) - \log x_0 + a \log z_0 &= 0 \\ (1-b)\log(x_0 + y_0 + z_0) - \log y_0 + b \log z_0 &= 0. \end{cases}$$

Let  $w_0 := 0.2405 \varphi(x_0, y_0, z_0)$ ; we compute  $x_0, y_0, z_0, w_0$  by approximation and obtain

$$w_0 - 0.9998 \in (0, 10^{-4}).$$

From Lemma 6 and Lemma 7,

$$\begin{aligned} \#W \cap \{1, \dots, N\} &\leq \text{constant} \cdot (\log N)^K \cdot e^{\varphi(x_0, y_0, z_0) \cdot 0.2405 \log N} \\ &\leq \text{constant} \cdot (\log N)^K N^{w_0} \end{aligned}$$

hence  $\#W \cap \{1, \dots, N\} \leq N^{0.9999}$  for  $N$  large enough.  $\square$

#### APPENDIX A. CLASSICAL BOUND FOR THE BINOMIAL COEFFICIENTS

**Lemma 8.** For any  $m, n \in \mathbb{N}$ ,  $\frac{(m+n)!}{m!n!} \leq \frac{(m+n)^{m+n}}{m^m n^n}$ .

*Proof.* Let  $f(m, n) = \frac{(m+n)!}{m!n!} \cdot \frac{m^m n^n}{(m+n)^{m+n}}$ , one has obviously  $f(m, 1) \leq 1$  and it remains to prove that  $f(m, n+1) \leq f(m, n)$ .

Let  $r = n+1$  and  $s = m+n+1$ , one has

$$\begin{aligned} \frac{f(m, n+1)}{f(m, n)} &= \frac{(m+n+1)!}{m!(n+1)!} \cdot \frac{m^m (n+1)^{n+1}}{(m+n+1)^{m+n+1}} \cdot \frac{m!n!}{(m+n)!} \cdot \frac{(m+n)^{m+n}}{m^m n^n} \\ &= \frac{(n+1)^n}{(m+n+1)^{m+n}} \cdot \frac{(m+n)^{m+n}}{n^n} \\ &= \left(\frac{r}{r-1}\right)^{r-1} \cdot \left(\frac{s-1}{s}\right)^{s-1}. \end{aligned}$$

Since  $r < s$  it remains to prove that the function  $g(x) = \left(\frac{x}{x-1}\right)^{x-1}$  is increasing: this holds because

$$\begin{aligned} (\log g(x))' &= \log\left(\frac{x}{x-1}\right) - \frac{1}{x} \\ &= -\log(1+t) + t \quad \text{with } t = -\frac{1}{x} \\ &\geq 0. \end{aligned}$$

$\square$

**Corollary 9.** For any  $m, n, p \in \mathbb{N}$ ,  $\frac{(m+n+p)!}{m!n!p!} \leq \frac{(m+n+p)^{m+n+p}}{m^m n^n p^p}$ .

*Proof.* Using Lemma 8,

$$\begin{aligned} \frac{(m+n+p)!}{m!n!p!} &= \frac{(m+n+p)!}{(m+n)!p!} \cdot \frac{(m+n)!}{m!n!} \\ &\leq \frac{(m+n+p)^{m+n+p}}{(m+n)^{m+n} p^p} \cdot \frac{(m+n)^{m+n}}{m^m n^n} = \frac{(m+n+p)^{m+n+p}}{m^m n^n p^p}. \end{aligned}$$

$\square$

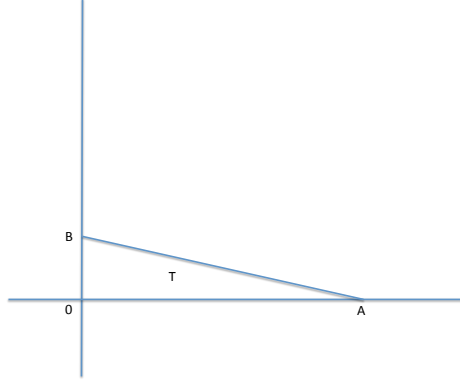
## APPENDIX B. STUDY OF A FUNCTION

**Lemma 10.** *Let  $a, b \in (0, 1]$ . The maximum of the function*

$$\varphi(x, y) := \log \left( \frac{(x + y + z)^{x+y+z}}{x^x y^y z^z} \right) \quad (\text{where } z = 1 - ax - by)$$

*is attained in the interior of the triangle  $T := \{(x, y) : x \geq 0, y \geq 0, z \geq 0\}$ .*

*Proof.*  $\varphi(x, y) = (x + y + z) \log(x + y + z) - x \log x - y \log y - z \log z$  is continuous on the closed triangle  $T$  whose vertices are the origin, the point  $A(\frac{1}{a}, 0)$  and the point  $B(0, \frac{1}{b})$ , hence it has a maximum on  $T$ .



Notice that  $x + y + z \neq 0$  on  $T$ . The partial derivative

$$\frac{\partial}{\partial x}(\varphi(x, y)) = (1 - a) \log(x + y + z) - \log x + a \log z$$

has limits  $+\infty$  when  $x \rightarrow 0$  and  $-\infty$  when  $x \rightarrow x_1$ , with  $x_1$  such that  $1 - ax_1 - by = 0$ , hence the map  $x \mapsto \varphi(x, y)$  increases at the neighborhood of 0 and decreases at the neighborhood of  $x_1$ , it has a maximum in the open interval  $(0, x_1)$ .

Similarly the map  $y \mapsto \varphi(x, y)$  has a maximum in the open interval  $(0, y_1)$ , with  $y_1$  such that  $1 - ax - by_1 = 0$ .

We deduce that the map  $(x, y) \mapsto \varphi(x, y)$  cannot have a maximum in the boundary of  $T$ .  $\square$

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